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Probing Behavior In Certain Optimal Perturbation Control Laws

by
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Research Department

JULY 1990

NAVAL WEAPONS CENTER
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Naval Weapons Center

FOREWORD

This report documents an extension of asymptotic approximations of optimal control laws to a wider class of cases than has been analyzed previously.

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(U) Asymptotic approximations of optimal control laws are established for a class of dynamic systems whose state components are either slowly varying (i.e., parameters) or can be tracked arbitrarily well by making the controller's measurements sufficiently accurate. These control optimization problems have the form assumed by higher-order descriptions of noised-induced perturbations from optimal nominal behavior in a more general type of control problem. When the measurement noise is small, bilinear measurement terms in the parameter and control variables can give rise to a rapidly fluctuating "probing" term in the optimal control. This term is the output of a high-frequency linear system driven by a product of the parameter covariance matrix and a Kalman-filter innovation vector. These results are derived with an approximate dynamic programming analysis where the neglected error terms are assumed to be sufficiently well behaved that all quantities that appear to be negligible actually are so in some appropriate sense; i.e., the results are only formal.

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CONTENTS

INTRODUCTION AND SUMMARY

Asymptotic approximations of optimal control laws are determined for a class of multivariate dynamic systems whose state components are either slowly varying (i.e., parameters) or can be tracked arbitrarily well by making the controller's state measurements sufficiently accurate. This control optimization problem would have the standard linear-quadratic-Gaussian form except for small terms in the measurements, that are bilinear in the control and parameter variables, and with respect to which the control law approximations are asymptotic. Also, the measurement noise is small in a certain relative sense, which gives this control problem special properties. The bilinear measurement terms give rise to a rapidly fluctuating term in the optimal control law—a linear function of the state of a multivariate linear system driven by a Kalman-filter innovation variable.

This rapidly fluctuating control component is essentially zero-mean, even conditioned on all but very recent data, and so has little current effect on the system dynamics. It is apparently an example of the probing phenomenon in optimal control laws identified by Feldbaum (Reference 1), which is current control effort expended to reduce uncertainty in the state variable (including the parameters here) in order to improve system performance in the future. A voluminous amount of literature exists on this general subject. In work related to the particular topic here, specialized methods have been used to obtain sharper results for a particular case related to homing missile guidance (Reference 2). Speyer and Hahn have derived similar asymptotic approximations of optimal control laws for systems with bilinear terms in the dynamics (Reference 3), but the "parameters" there are not static or slowly varying and the measurement noise is not small.

The results here are obtained from an approximate dynamic programming analysis in which the state variable includes the departures of certain Kalman-filter covariance quantities from their local averages, and the analysis uses the fact that these departures are small because of the small measurement noise. It is then shown how these results apply to other cases that arise from analyses of noise-induced departures from nominal behavior in a more general class of optimal control problems. Typically, these other cases have (relatively small) quadratic terms in the system dynamics and state measurements, and cubic terms in the performance criterion. The analysis here is not at a mathematically rigorous level, although the constructions developed might be useful in a more ambitious treatment of that sort. Expressions denoting ordinary differential equations with white noise terms should be understood as the formally corresponding stochastic differential equations in the Ito sense (Reference 4 and 5) if a rigorous interpretation is desired.

Unless otherwise stated, lower case letters denote (real finite-dimensional) column vectors and scalars. Matrices are denoted by capital Roman letters. A^T denotes the transpose of a matrix A, and tr(A) its trace if A is square. It will also be convenient to make use of three-way matrices, which are always denoted by capital Greek letters here. For continuity of notation, the following definitions are adopted for such a three-way matrix Γ, with vector x and matrices A and B of compatible dimensions and with repeated indices denoting summation:

$(\Gamma x)_{ij} = \Gamma_{ij\sigma} x_\sigma$ (matrix)

$(A x^T)_{ijk} = A_{ij} x_k$ (three-way matrix)

$(A\Gamma)_{ijk} = A_{i\sigma}\Gamma_{\sigma jk}$ (three-way matrix)

$(\Gamma B)_{ijk} = \Gamma_{ij\sigma} B_{\sigma k}$ (three-way matrix)

$(\Gamma')_{ijk} = \Gamma_{jki}$ and $(\Gamma^T)_{ijk} = \Gamma_{kji}$ (three-way matrix)

$[tr(\Gamma)]_i = \Gamma_{\sigma i \sigma}$ (column vector, when applicable).

With these definitions, the expression $FA\Gamma BDxx^T$ is fully associative. Many other consequences are obvious. Some useful but less obvious properties are

$$tr(\Gamma' x) = [tr(\Gamma)]^T x$$

$$A\ tr(\Gamma') = tr[(A\Gamma)']$$

$$tr(A\Gamma) = tr(\Gamma A)$$

$$(\Gamma B)' = B^T \Gamma' \text{ and } (A\Gamma)'' = \Gamma'' A^T$$

$$(A\Gamma B)^T = B^T G^T A^T$$

$$(\Gamma' x)A^T = (A\Gamma)' x \text{ and } (\Gamma'' x)B = (\Gamma B)'' x.$$

PROBLEM STATEMENT

The more limited case actually analyzed here is that of a multivariate system with dynamics of the form

$$\dot{x} = Fx + Gu + w \qquad (1)$$

$$\dot{\theta} = h\, w_2, \qquad (2)$$

a controller of which receives a vector measurement of the form

$$z = Hx + h\, H\, tr(\Gamma''\theta u^T) + v \tag{3}$$

and selects the control vector u at each time instant $t \geq 0$ (with the convention that Equation 3 is evaluated slightly before it is used to change u). The initial state partitions $x(0)$ and $\theta(0)$ are statistically independent a priori with Normal ($\hat{x}_0$, P_0) and Normal (0, L_0) distributions respectively. Zero-mean Gaussian white noise processes w_1, w_2, and v are independent with covariance parameters Q, D and R/m^2, respectively. Positive scalars m and h and are considered parameters of the problem, with $1 \ll m \ll 1/h$ and usually further restrictions. The objective is to find a control law that minimizes a scalar performance criterion of the form

$$J = \frac{1}{2}\, \mathcal{E}\left[x_f^T S_f x_f + \int_0^{t_f} (x^T A x + u^T B u + 2x^T Z u) dt \right], \tag{4}$$

where $\mathcal{E}$ denotes prior expectation and t_f is some specified terminal time. As usual, a control law is defined as a decision rule that, for each t in $[0, t_f)$, specifies the current control u(t) as a function of the current measurement history $\{(z(s), s) : 0 \leq s < t\}$. A, B, and S_f are symmetric with A, S_f, P_0, L_0, Q, and D positive-semidefinite and B and R positive-definite. In the context of sufficiently small h, the components of all the matrices as well as R^{-1} and B^{-1} are of order unity. These matrices may be time-varying but only with time-derivatives that are also of order unity in this sense.

Also, some further structural properties are assumed for the limiting form of this problem when h = 0 (in which case θ is irrelevant). First, it is assumed that (F, G) is stabilizable and (F, H) is observable at each time instant. To specify the other properties and also for future reference, it is convenient to define V(t) (a normalized covariance matrix of x for this limiting case) and a corresponding transition matrix T by

$$\dot{V} = FV + VF^T + m\,(Q - VH^T R^{-1} HV);\; V(0) = mP_0 \tag{5}$$

and

$$\frac{\partial T\,(s,t)}{\partial s} = [F - mVH^T R^{-1} H]\,(s)\; T(s,t);\; T(t,t) = I. \tag{6}$$

Other assumptions are made: with m considered as a problem-formulation parameter, there exists a positive $\tau(m)$ (a "settling time") such that

$$\lim_{m\to\infty} \tau = 0 \text{ and } \lim_{m\to\infty} m\tau \geq 1,$$

and for all sufficiently large m and $t \gg \tau$,

(1) V is of order $m\tau$

(2) T(s,t) is of order $m\tau e^{-(s-t)/\tau}$ } for $s \geq t$.

(3) $T(s,t)VH^T$ is of order $e^{-(s-t)/\tau}$ } for $s \geq t$.

(4) If U and X are symmetric matrix time functions of order unity with X positive-semidefinite and $\dot{U}$ and $\dot{X}$ also of order unity, and if K and q are the matrix time function and vector random process defined by

$$-\dot{K} = K(F - mVH^TR^{-1}H) + (F^T - mH^TR^{-1}HV)K + U;\; K(t_f) = 0$$

and

$$\dot{q} = (F - mVH^TR^{-1}H)q + mw_3;\; q(0) = 0,$$

where

w_3 is zero-mean Gaussian white noise with covariance parameter X, then

(a) $T^T(s,t)KVH^T$ is of order $m\tau^2$ for $s \geq t$

and

(b) $HV\dot{K}q$ is of order $m^2\tau^{5/2}$

(except perhaps for a negligibly improbable set of realizations of q).

Because T(s,t) decays rapidly with increasing $(s - t) > 0$, the variations of V(s), H(s), and K(s) are negligible until T is essentially zero; thus, (3) and (4a) will also hold when V, H, and K are evaluated at s instead of t. Also, since V/m is the error covariance matrix for the Kalman-filter estimate of x, these assumptions imply that the entire vector x can be tracked arbitrarily closely if the measurement noise can be made small enough (by (1), (2), and $\tau \to 0$ as $m \to \infty$).

Finding an optimal control law exactly is very difficult for this case, and we seek only an approximation that is asymptotically accurate to order $h^2m^2\tau^{5/2}$ for small h and $h^2m^6\tau^{11/2} \ll 1$ (except perhaps for a set of measurement histories of negligible probability). However, since τ can be as large as $m^{-1/n}$ when x is n-dimensional (see next section), control terms of this magnitude can be large compared to h^2 under these conditions if $n \geq 2$, which is necessary for their significance in the type of applications described in the last section. The treatment of the problem here is further limited to finding the control law corresponding to a cost-to-go function that has the formal appearance of satisfying the Bellman equation for Equations 1 through 4 to order $h^2 m^2 \tau^{5/2}$. This control law would be the desired approximation if the equations involved in the analysis are well-posed—and the formally negligible terms are indeed so in some appropriate sense—but a rigorous verification of these conditions is beyond the scope of this report. In this sense, the result obtained below is only a plausible candidate for the control-law approximation being sought.

HEURISTIC MOTIVATION FOR RESTRICTIONS

Although not readily apparent, the extra restrictions (1) through (4) in the problem formulation are actually motivated by the prototypical case of an n-dimensional system with $\dot{x}_i = x_{i+1}$, $i = 1, \ldots, n-1$, $\dot{x}_n$ = order-unity noise w, and scalar measurement of x_1 only. One way of estimating the current value of this state vector for $m \gg 1$ (i.e., small measurement noise) is to partition a short preceding time interval, say of length τ, into n equal subintervals, estimate the value of x_1 at the midpoint of each subinterval as the average value of z observed during it, and then estimate $x_{i+1} (= x^{(i)})$, $i = 1, \ldots, n-1$ by taking an ith-order difference of these x_1-estimates and dividing by the appropriate multiple of τ^i. By standard results, the error variance in the x_1 estimates in the absence of the process noise w is $m^{-2}\tau^{-1}$ and that of the x_i-estimate is of order $m^{-2}\tau^{1-2i}$. The error variance in the estimate of x_i would therefore decrease with increasing τ until the variance of the change in x_i produced by the process noise over that interval (which increases with τ) is of the same order of magnitude. For each i, this happens when τ is of order $m^{-1/n}$, because x_i is the $(n - i + 1)$st integral of w, whose variance over an interval of length τ can be obtained by standard methods and is of order $\tau^{2(n-i)+1}$. For any given n, $m^{-1/n} \rightarrow 0$ as $m \rightarrow \infty$, so an optimal τ for this kind of estimation would indeed be short and have this order of magnitude for sufficiently large m (i.e., small enough measurement noise). The corresponding error variance in the estimate of x_i would then be of order τ^{n+1-2i}/m. Since the $(n \times n)$ covariance matrix P for these errors must be positive-semidefinite, component $P_{i,j}$ would be of order $\tau^{n+1-i-j}/m$.

The basic premise here is that the Kalman filter for such a system should behave in the same way to an order-of-magnitude approximation (this has been verified for some special cases). In the notation of the preceding section for such a system, the properties assumed there can then be established by concluding in turn that:

(1) $V_{i,j}$ is of order τ^{1-i-j}/m; because $V = mP$

(2) $(VH^T)_i$ is of order τ^{-i}/m; because $H = [1, 0, ..., 0]$ in this case

(3) $T_{i,j}(t,s)$ is $\begin{cases} \text{of order } \tau^{j-i} \text{ for positive } t - s \text{ of order } \tau \\ \text{negligible for } \tau - s \gg \tau \end{cases}$

in order for the transition-matrix solution (e.g., Reference 6) of the Kalman filter's differential equation for the conditional mean to operate (as a convolution) on the measurement history in the manner described above.

(4) $K_{i,j}$ is of order $m^2\tau^{1+i+j}$; from the transition-matrix solution of $\dot{K}$, (3), and the fact that m is of order τ^{-n} in this case.

(5) $var(q_i)$ is of order $m^2\tau^{3-2i}$; from (3) in the transition-matrix solution of the usual differential equation for the covariance matrix of q, and the fact that $\tau \ll 1$ and m is of order τ^{-n}.

(6) q_i is of order $m\tau^{3/2-i}$; by (5), the Chebychev inequality, and the fact that q is zero-mean.

(7) V is of order $m\tau$; from (1) and the fact that $\tau \ll 1$ and m is of order τ^{-n}.

(8) $T(t,s)$ is of order $m\tau e^{(s-t)/\tau}$ for $t \geq s$; by (3) and the fact that $\tau \ll 1$ and m is of order τ^{-n}.

(9) $T(t,s)VH^T$ is of order $e^{(s-t)/\tau}$; from (2), (3), and the fact that $\tau \ll 1$ and m is of order τ^{-n}.

(10) $T^T(t,s)KVH^T$ is of order $m\tau^2$; from (2) through (4) and the fact that $\tau \ll 1$ and m is of order τ^{-n}.

(11) HVKq is of order $m^2\tau^{5/2}$; from (2), (4), (6), and the fact that m is of order τ^{-n}.

In this case, t would be a meaningful "settling time" for this Kalman filter. *Although x can be tracked arbitrarily well* with sufficiently accurate measurements (of x_1 alone in this instance), an interesting point in this connection is that if $\dot{x}_k$ were also unit-intensity process noise for some $k < n$, then the Kalman filter's effective "data window" for estimating $x_{k+1}, ..., x_n$, and the variance of the estimation errors for these state components, would remain of order unity as $m \rightarrow \infty$. The limiting factor in estimating these components would then be the ratio of the process noise intensities to each other, and the entire state vector could no longer be tracked arbitrarily well just by reducing the measurement noise even though the system would still be observable.

Finally, it is reasonable to hope that the properties assumed in the problem formulation would also be characteristic of a considerably wider class of linear observation systems whose state can be tracked arbitrarily well with small enough measurement noise. The reasoning is that the tracking accuracy in such cases would often be limited by some similarly configured subset of the state components, one of which is driven by process noise and is a higher derivative of another that is measured directly.

STATE ESTIMATION

If the conditional mean and covariance matrix for the current composite state (x,θ), given the currently available measurements, are partitioned in the obvious way as

$$\begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} P & E \\ E^T & L \end{bmatrix},$$

then it follows from standard Kalman filtering results (Reference 7) that

$$\dot{\hat{x}} = F\hat{x} + Gu + m^2[P + h(\Gamma^T E^T)'u]H^T R^{-1}(z - \hat{z});\ \hat{x}(0) = \hat{x}_0 \tag{7}$$

$$\dot{\hat{\theta}} = m^2[E^T + h(\Gamma^T L)'u]H^T R^{-1}(z - \hat{z});\ \hat{\theta}(0) = \hat{\theta}_0 \tag{8}$$

$$\dot{P} = FP + PF^T + Q - m^2[P + h(\Gamma^T E^T)'u]H^T R^{-1} H[P + h(E\Gamma)'u]; P(0) = P_0 \tag{9}$$

$$\dot{E} = FE - m^2[P + h(\Gamma^T E^T)'u]H^T R^{-1} H[E + h(L\Gamma)'u];\ E(0) = 0 \tag{10}$$

$$\dot{L} = h^2 D - m^2[E^T + h(\Gamma^T L')u]H^T R^{-1} H[E + h(L\Gamma)'u];\ L(0) = L_0, \tag{11}$$

where

$$\hat{z} = H[\hat{x} + h\ \mathrm{tr}(\Gamma''\hat{\theta}u^T)]. \tag{12}$$

It will be convenient to define the quantities

$$M = \frac{m}{h}[E + h(L\Gamma)'u] \tag{13}$$

and

$$N = (mP - V)/h^2 + mu^T\Gamma'' L\Gamma^{T'} u, \tag{14}$$

where V(t) is as defined by Equation 5, and the normalized Kalman-filter innovation variable

$$\xi = m(z - \hat{z}). \tag{15}$$

Differentiating Equations 13 and 14 and substituting from Equations 7 through 12 then show eventually that

$$\dot{\hat{x}} = F\hat{x} + Gu + [V + h^2N + h^2(\Gamma^TM^T)'u]H^TR^{-1}\xi, \tag{16}$$

$$\begin{aligned}\dot{M} = {} & (F - mVH^TR^{-1}H)M + m(L\Gamma)'\dot{u} + m[L(\dot{\Gamma} - \Gamma F^T) \\ & + h^2(D - M^TH^TR^{-1}HM)\Gamma]'u - mh^2[N + (\Gamma^TM^T)'u]H^TR^{-1}HM,\end{aligned} \tag{17}$$

$$\begin{aligned}\dot{N} = {} & (F - mVH^TR^{-1}H)N + N(F^T - mH^TR^{-1}HV) - m(VH^TR^{-1}H\Gamma^TM^T \\ & + M\Gamma H^TR^{-1}HV)'u - mh^2[N + (\Gamma^TM^T)'u]H^TR^{-1}H[N + (M\Gamma)'u] \\ & + m[F(u^T\Gamma''L\Gamma^{T'}u) + (u^T\Gamma''L\Gamma^{T'}u)F^T - \frac{d}{dt}(u^T\Gamma''L\Gamma^{T'}u)],\end{aligned} \tag{18}$$

and

$$\dot{L} = h^2(D - M^TH^TR^{-1}HM). \tag{19}$$

L clearly remains positive-semidefinite and of order unity over time intervals that are short compared to $1/h^2$. Also, ξ can be treated as a zero-mean Gaussian white noise process with covariance parameter R(t) in determining the statistical behavior of the above Kalman-filter variables (Reference 4).

Controls with Wiener-process components will be used below; they are not actually differentiable and so should really be interpreted as smooth approximations of the Wiener process used formally. Such approximations would be unavoidable in practice anyway and presumably could be made in a way that amounts to local averaging over a short enough time interval that the difference would be negligible to the order of accuracy retained in the analysis.

APPROXIMATE ESTIMATOR BEHAVIOR IN A RESTRICTED CONTEXT

If h = 0, it is a standard result that the optimal control law is

$$u = -W\hat{x}, \tag{20}$$

where $W = B^{-1}(Z^T + G^TS)$ with $S(t)$ the solution of

$$-\dot{S} = SF + F^TS + A - (SG + Z)B^{-1}(Z^T + G^TS);\ S(t_f) = S_f. \tag{21}$$

Since we are only concerned with small h here, we consider control laws of the form

$$u = -W\hat{x} + \eta, \tag{22}$$

under the further restriction that $h^2m^6\tau^{11/2} \ll 1$, such that η is small compared to unity in that context. With respect to sufficiently small h, it is clear that the assumed properties of the matrices defining the problem, including the stabilizability of (F, G) and the observability of (F, H), imply that W and $\dot{W}$ can be considered of order unity and that x and $\hat{x}$ are also of order unity by this standard (except perhaps for a set of realizations of negligible probability) when the control law of Equation 20 is used and h = 0.

Differentiating Equation 22 and using Equation 16 shows that

$$\begin{aligned}\dot{u} = -(\dot{W} + WF + WGW)\hat{x} - WG\eta - W[V + h^2N + h^2(\Gamma^TM^T)' \\ (\eta - W\hat{x})]H^TR^{-1}\xi\end{aligned} \tag{23}$$

for such a control law if the perturbation control η is small and smooth enough that $\dot{\eta}$ is negligible in the following (which it will be for the optimal control law approximation derived below). We now assume this to be the case and further that M and N are of respective orders $m^2\tau^{3/2}$ and $m^3\tau^{5/2}$ with the autocovariance functions of the M-components being negligible for time differences that are large compared to τ. Then it follows from Equation 19 that order-unity changes in the L-components behave basically as sums of $1/(h^2m^4\tau^4)$ independent random increments, each with a variance of order $h^4m^8\tau^8$. Hence the variance of such a sum is of order $h^2m^4\tau^4$, which is small compared to unity under the restrictions of interest here because $1 \gg h^2m^6\tau^{11/2} = (h^2m^4\tau^4)(m\tau)^{3/2}\sqrt{m}$. This means by the Chebychev inequality that the difference between L and its prior expected value $\bar{L}(t)$ is small compared to unity. In consequence, the substitution from Equations 22 and 23 for u and $\dot{u}$ in Equations 16 through 18 gives

$$\dot{\hat{x}} = (F - GW)\hat{x} + VH^TR^{-1}\xi, \tag{24}$$

$$\dot{M} = (F - mVH^TR^{-1}H)M - m(\bar{L}\Gamma)'WVH^TR^{-1}\xi, \tag{25}$$

and

$$\dot{N} = (F - mVH^TR^{-1}H)N + N(F^T - m^TR^{-1}HV) +$$
$$m(VH^TR^{-1}H\Gamma^TM^T + M\Gamma H^TR^{-1}HV)W\hat{x}, \tag{26}$$

except for higher-order terms, when M and N are of the assumed orders of magnitude and $h^2m^6\tau^{11/2} \ll 1$.

Since R and R^{-1} are both of order unity and since Equation 24 is the equation for $\dot{\hat{x}}$ when h = 0, the components of the product VH^T must be of order unity because $\hat{x}$ is of order unity then.

Let q denote an arbitrary column of M and k its index. Then from Equation 25,

$$\dot{q} = (F - mVH^TR^{-1}H)q - mUWVH^TR^{-1}\xi;\ q(0) = 0, \tag{27}$$

where U is the matrix such that $U_{ij} = (\bar{L}\Gamma)_{kji}$. Let C(t) denote the prior covariance matrix of q(t). It follows from Equation 27 and standard results that

$$\dot{C} = (F - mVH^TR^{-1}H)C + C(F^T - mH^TR^{-1}HV) + m^2UWVH^TR^{-1}HVW^TU^T;$$
$$C(0) = 0$$

and that

$$C(t) = m^2\int_0^t T(t,s)[UWVH^TR^{-1}HVW^TU](s)T^T(t,s)ds.$$

From the properties assumed for the transition matrix T and the fact that VH^T is of order unity, C is of order $m^4\tau^3$. By the Chebychev inequality, therefore, the components of any column of M and, hence, any component of M itself would be of order $m^2\tau^{3/2}$ in this approximation, except perhaps for a set of realizations of negligible probability.

Furthermore if $\bar{C}(t_2,t_1)$ is used to denote the prior expectation of $q(t_1)q^T(t_2)$ for $t_2 \geq t_1$, it is a standard result that

$$\frac{\partial\bar{C}(t_2,t_1)}{\partial t_2} = [F - mVH^TR^{-1}H](t_2)\,\bar{C}(t_2,t_1);\ \bar{C}(t_1,t_1) = C(t_1).$$

and, hence, that $\bar{C}(t_2,t_1) = T(t_2,t_1)C(t_1)$. Since $T(t_2,t_1)$ decays exponentially as $(t_2 - t_1)/\tau$ by assumption, $\bar{C}(t_2,t_1)$ is indeed negligible for $t_2 - t_1 \gg \tau$.

Now define $X(t) = T(s,t)M(t)$ for $t \leq s$, so

$$\dot{X} = \frac{\partial T(s,t)}{\partial t} M(t) + T(s,t)\dot{M}(t).$$

From standard results for transition matrix derivatives and the above approximation for $\dot{M}$,

$$-\dot{X} = mT(s,t)(L\Gamma)' WVH^T R^{-1}\xi;\ X(s) = M(s)$$

except for higher-order terms. Integration in reverse time from $t = s$ gives $X(t) - M(t)$ as a zero-mean random variable with component variances of order $m^4\tau^3$ because of the order of magnitude assumed for T. Since M(t) itself is of order $m^2\tau^{3/2}$, so is T(s,t)M(t) for $t \leq s$ by the Chebychev inequality.

With routine matrix manipulation, the solution to the approximate equation for $\dot{N}$ can be expressed as

$$N(t) = m\int_0^t \{T(t,s)[\Omega^T(s)M^T(s) + M(s)\Omega(s)]T^T(t,s)\}' W(s)\hat{x}(s)ds,$$

where Ω denotes $\Gamma H^T R^{-1} HV$. Because of the preceding order-of-magnitude results, this shows that N is of order $m^3\tau^{5/2}$.

Under the restrictions specified above, the Kalman filter equations (expressed in terms of V, M, N, L, $\hat{x}$, and $\hat{\theta}$) are therefore formally consistent with the orders of magnitude and correlation-function behavior assumed for M and N, and verify them under the assumption that the full equation system is well posed in some approximate sense, and under the further assumption that $\dot{\eta}$ would be negligible in the preceding derivations. As further consequences, moreover, VH^T is of order unity, T(s,t)M(t) is of order $m^2\tau^{3/2}$ for $s \geq t$, and the difference between L(t) and its prior expected value $\bar{L}(t)$ is small compared to unity under these conditions.

CONTROL OPTIMIZATION

For S(t) and u as specified by Equations 21 and 22, the problem here reduces to that of finding an optimal control law for the perturbation control η to which we seek only an asymptotic approximation. An optimal expected cost-to-go function can be defined consistently in terms of time and the conditional distribution of x and θ (Reference 8). The Principle of Optimality of dynamic programming can then be applied in the usual way (Reference 9) to derive a Bellman equation for this function, the solution of which equation specifies the optimal control law for η.

Following this approach, we consider a possible cost-to-go function of the form

$$J = \frac{1}{2}\hat{x}^T S\hat{x} + \frac{h^2}{2m} \operatorname{tr}[(S + Y)N] + h^2\hat{x}^T \operatorname{tr}(\Lambda'' M) + f(t), \tag{28}$$

where S(t) is as defined by Equation 21, and is hence symmetric and of order unity, and where Y and Λ are functions of t only, with Y symmetric and of order $m^2\tau^3$, and with Λ of order $m\tau^3$. Retaining accuracy only to orders $h^2m^2\tau^{5/2}$ and $h^2m^2\tau^{5/2}\gamma$ (where γ denotes any product of η components) and using Equations 16 through 18 (with Equations 22 and 23 substituted for u and $\dot{u}$) to evaluate the required conditional expected increments (and products thereof) in $\hat{x}$, M and N result in a Bellman equation of the form

$$\left.\begin{aligned} -\frac{\partial J}{\partial t} &= \min_{\eta} \rho(t, \hat{x}, M, N, \eta) \\ J(t_f, \ldots) &= \frac{1}{2}\{\hat{x}^T S_f\hat{x} + \operatorname{tr}[S_f(V_f + h^2N)]/m\} \end{aligned}\right\}, \tag{29}$$

where ρ is a certain rather lengthy expression, if $h^2m^6\tau^{11/2} \ll 1 \ll m^6\tau^{5/2}$. This derivation also uses the fact that L can be replaced by its prior expected value $\bar{L}$ in Equations 17 and 18 in this context. The minimizing η in Equation 29 occurs when the η-derivative of the expression ρ is zero, which is easily shown to be when

$$\eta = h^2B^{-1}\operatorname{tr}\{[\Gamma H^TR^{-1}HVY - (\Lambda G)'']M\}. \tag{30}$$

Substituting from Equation 30 for η in Equation 29 and collecting terms in like powers of the a priori random variables $\hat{x}$, M, and N then show that J of Equation 28 formally satisfies the Bellman equation to order $h^2m^2\tau^{5/2}$, for $h^2m^6\tau^{11/2} \ll 1 \ll m^6\tau^{5/2}$ and S(t) the solution of Equation 21, if

$$\begin{aligned} -\dot{Y} = {} & Y(F - mVH^TR^{-1}H) + (F^T - mH^TR^{-1}HV)Y + \\ & (SG + Z)B^{-1}(Z^T + G^TS);\ Y(t_f) = 0 \end{aligned} \tag{31}$$

$$\begin{aligned} -\dot{\Lambda} = {} & \Lambda[F - GB^{-1}(Z^T + G^TS)] + (F^T - mH^TR^{-1}HV)\Lambda + \\ & YVH^TR^{-1}H\Gamma' B^{-1}(Z^T + G^TS);\ \Lambda(t_f) = 0 \end{aligned} \tag{32}$$

$$f(t) = \int_t^{t_f} g(s)ds,$$

if the resulting values of Y and Λ have the assumed orders of magnitude and if the corresponding perturbation control of Equation 30 is such that the contributions of $\dot{\eta}$ would be insignificant in the computation of $\dot{N}$ and $\dot{M}$.

Now by standard properties of transition matrices,

$$Y(t) = \int_t^{t_f} T^T(\alpha, t)[SG + Z)B^{-1}(Z^T + G^TS)](\alpha)\, T(\alpha, t)\, d\alpha.$$

Since $T(\alpha, t)$ decays to zero by assumption over an α-interval of order τ and has maximum component magnitudes of order $m\tau$, this integration is effectively over an interval of order τ and Y is therefore of order $m^2\tau^3$, as assumed. Also, for the transition matrix T_2 defined by

$$\frac{\partial T_2(\alpha,t)}{\partial \alpha} = [F - (SG + Z)B^{-1}G^T](\alpha)\, T_2(\alpha, t);\ T_2(t, t) = I,$$

it is straightforward to verify that

$$\Lambda(t) = \int_t^{t_f} T^T(\alpha, t)[YVH^TR^{-1}H\Gamma' B^{-1}(Z^T + G^TS)](\alpha)\, T_2(\alpha, t)\, d\alpha.$$

Since $T_2(\alpha, t)$ is order unity and this integration is also effectively over an interval of order τ, Λ is of order $m\tau^3$, as assumed, by the unity order of magnitude postulated for matrix products of which T^TYVH^T is an example. Finally, the time derivative $\dot{\eta}$ of the derived optimal perturbation control will, as a result, be dominated by terms of orders $h^2m^3\tau^3M$, $h^2m^4\tau^5M$, $h^2m^2\tau^3\dot{M}$, and $h^2m^3\tau^5\dot{M}$. Such terms, as was assumed, would make no significant contribution as a part of $\dot{u}$ in the differential Equations 17 and 18 for M and N if $h^2m^6\tau^{11/2} \ll 1$.

To summarize, it has been shown at least formally that the optimal control law is

$$u = -B^{-1}(Z^T + G^TS)\hat{x} + \eta$$

to order $h^2m^2\tau^{5/2}$ if $h^2m^6\tau^{11/2} \ll 1 \ll m^6\tau^{5/2}$, where S(t) is given by Equation 21, η is given by Equation 30 with V(t), Y(t), and $\Lambda(t)$ as given by Equations 5, 31, and 32, and where $\hat{x}$ and M are generated from the incoming measurements z by the Kalman filter of Equations 7 through 12 and

$$\begin{aligned}\dot{M} &= (F - mVH^TR^{-1}H)M - m^2(L\Gamma)'B^{-1}(Z^T + G^TS)VH^TR^{-1}(z - \hat{z});\\ M(0) &= 0.\end{aligned} \tag{33}$$

The last equation is a sufficiently accurate truncation of Equation 17 with substitution from Equations 22 and 23 for u and $\dot{u}$.

APPLICATIONS IN A MORE GENERAL CONTEXT

The preceding results also have applications to a wider class of optimal control problems, with multivariate dynamics of the general form

$$\begin{cases} \dot{x} = f(x, \theta, u, t, v) & \text{(motion-state)} \\ \dot{\theta} = 0 & \text{(parameters)} \\ x(t_0),\ \theta \text{ Normal and independent a priori with } \mathcal{E}(\theta) = 0, & \end{cases}$$

state measurements of the form

$$\begin{cases} z = g(x, \theta, u, t, v) \\ w,\ v \text{ independent white noise processes,} \end{cases}$$

and performance criterion (to be minimized) of the form

$$J = \mathcal{E}\left[\psi(x_f, \theta) + \int_t^{t_f} \lambda(x, \theta, u, t)dt\right],$$

where t_0 and t_f are specified a priori. Corresponding to such a problem, there is the nominal control problem (easier to solve because it is deterministic) of finding $\bar{u}(t)$ to minimize

$$\bar{J} = \psi(\bar{x}_f, t_f) + \int_t^{t_f} \lambda[\bar{x}(t), 0, \bar{u}(t), t]dt$$

for

$$\dot{\bar{x}} = f(\bar{x}, 0, \bar{u}, t, 0);\ \bar{x}(t_0) = \text{prior mean of } x(t_0).$$

Also, a nominal measurement history $\bar{z}(t)$ can be defined as

$$\bar{z}(t) = g[\bar{x}(t), 0, \bar{u}(t), t, 0]$$

when the minimizing $\bar{u}(t)$ is used.

Once $\bar{x}(t)$, $\bar{u}(t)$, and $\bar{z}(t)$ have been found for the solution of the nominal control problem, the original problem can be solved by adding to $\bar{u}(t)$ the perturbation control law for $u-\bar{u}$ (or δu), which minimizes $J-\bar{J}$ (or δJ). Taylor-series expansions of f, g, ψ, and λ can typically be used to express the dynamics, measurements, and criterion of this optimal perturbation control problem in terms of the perturbation variables δx, δu, δz, θ, w, and v (and, of course, the predetermined time functions $\bar{x}$, $\bar{u}$, and $\bar{z}$). Treating the perturbation variables as small and truncating the Taylor-series expansions at the lowest nontrivial order of accuracy lead to a problem of the familiar linear-quadratic-Gaussian form for the optimal perturbation control law. The solution to this problem is well known and can be added to $\bar{u}$ as a first approximation of the optimal control law for the original problem.

If the Taylor-series approximations are carried out to one higher order of asymptotic accuracy in this perturbation control problem, the effect is typically to introduce quadratic terms in the dynamics and measurements, and cubic terms in the criterion. Such a problem can often be rescaled so that the perturbation variables are of order unity and the coefficients of the added higher-degree terms become the quantities that are relatively small, say of order h. For cases formulated so that the parameter vector θ does not enter the measurement equation in the linear-quadratic-Gaussian approximation, possibly by adjoining the observable subspace of the original "parameters" to x, and with $\dim(x) \geq \dim(z)$, this higher-order problem typically can be transformed further to one with dynamics, measurements, and criterion of the form

$$\dot{y} = \bar{F}y + \bar{G}u + \mathrm{tr}(\Delta yy^T) + 2\mathrm{tr}(\boldsymbol{\Theta}'' yu^T) + \mathrm{GWN}(\bar{Q}), \tag{34}$$

$$z = \bar{H}y + \mathrm{tr}(\Phi yy^T) + 2\bar{H}\,\mathrm{tr}(\Omega yu^T) + \mathrm{GWN}(\bar{R}), \tag{35}$$

and

$$\begin{aligned} J = \mathcal{E}\{&\frac{1}{2} y_f^T \bar{S} y_f + \frac{1}{3} y_f^T \mathrm{tr}(\Pi_f y_f y_f^T) + \int_{t_0}^{t_f} [c^T y + y^T \bar{Z} u \\ &+ \frac{1}{2}(y^T \bar{A} y + u^T B u) + y^T \mathrm{tr}(\Psi uu^T) + u^T \mathrm{tr}(\Upsilon yy^T) \\ &+ \frac{1}{3} y^T \mathrm{tr}(\Sigma yy^T) + \frac{1}{3} u^T \mathrm{tr}(\Xi uu^T)]dt\}, \end{aligned} \tag{36}$$

where c and the three-way matrices are of order h and B^{-1} and all other quantities are of order unity, where u, z, and J are now used in place of δu, δz, and δJ, where "GWN" denotes Gaussian white noise with the indicated covariance parameter, where y denotes the composite vector $\begin{bmatrix} \delta x \\ \theta \end{bmatrix}$, and where $\bar{F}, \bar{G}, \bar{H}, \bar{Q}, \bar{S}, \bar{Z}$, and $\bar{A}$ have the forms

$$\bar{F} = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}, \bar{G} = \begin{bmatrix} G \\ 0 \end{bmatrix}, \bar{H} = [H \vdots 0], \bar{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bar{S} = \begin{bmatrix} S_f & 0 \\ 0 & 0 \end{bmatrix}, \bar{Z} = \begin{bmatrix} Z \\ 0 \end{bmatrix}, \text{ and } \bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

for this partitioning of y. If R^{-1} is also of order unity, the optimal control law for this problem can be approximated up to terms which are asymptotically of order h^2 (see appendix for details), and this approximations retains this accuracy as $\bar{R} \to 0$ if $\Omega = 0$ (Reference 10). Furthermore, the Taylor-series truncations in the problem formulation don't justify order-h^2 accuracy in the solution anyway.

For nonzero Ω, however, the analysis of the preceding sections shows that if $\bar{R} = R/m^2$, and R, F, and H have the properties assumed there with $h^2 m^6 \tau^{11/2} \ll 1$, then the optimal control law can also have another component (Equation 30) of order $h^2 m^2 \tau^{5/2}$. This latter component can be large compared to h^2 under these conditions and is therefore significant at the level of accuracy of the problem formulation. In this case, this extra control component can simply be added to the (order-h) control law approximation of the appendix to approximate the optimal control law to order $h^2 m^2 \tau^{5/2}$. Absorbing h and m into the definitions of Γ and R in earlier sections allows this added control component to be expressed as

$$\eta = B^{-1} \mathrm{tr}\{[\bar{\Gamma} \bar{H}^T \bar{R}^{-1} \bar{H} \bar{P} Y - (\bar{\Lambda} \bar{G})''] \bar{M}\}, \tag{37}$$

where $\bar{\Gamma}$ is the $p \times m \times n$ three-way matrix such that

$$\bar{\Gamma}_{i,j,k} = \Omega_{j,k,i+n} \qquad \left\{ \begin{aligned} n &= \dim(x) \\ m &= \dim(u) \\ p &= \dim(\theta) \end{aligned} \right. ,$$

where $\bar{P}(t)$, $Y(t)$, and $\bar{\Lambda}$ are the solutions of

$$\dot{\bar{P}} = \bar{F}\bar{P} + \bar{P}\bar{F}^T + Q - \bar{P}\bar{H}^T\bar{R}^{-1}\bar{H}\bar{P};\ \bar{P}(t_0) = \mathrm{var}(x_0) \tag{38}$$

$$-\dot{Y} = Y(F - \bar{P}H^T\bar{R}^{-1}H) + (F^T + H^T\bar{R}^{-1}H\bar{P})Y + (SG + Z)B^{-1}(Z^T + G^TS);$$
$$Y(t_f) = 0 \tag{39}$$

$$-\dot{\bar{\Lambda}} = \bar{\Lambda}[F - GB^{-1}(Z^T + G^TS)] + (F^T + H^T\bar{R}^{-1}H\bar{P})\,\bar{\Lambda} +$$
$$Y\bar{P}H^T\bar{R}^{-1}H\bar{\Gamma}'B^{-1}(Z^T + G^TS);\ \Lambda(t_f) = 0, \tag{40}$$

with S(t) as defined by Equation 21, and where $\bar{M}$ is generated by

$$\dot{\bar{M}} = (F - \bar{P}H^T\bar{R}^{-1}H)\bar{M} - (L\bar{\Gamma})'B^{-1}(Z^T + G^TS)\bar{P}H^T\bar{R}^{-1}(z - H\hat{x});$$
$$\bar{M}(t_0) = 0 \tag{41}$$

with L denoting the conditional covariance of θ (as computed in any of the preceding approximations of the optimal perturbation control law).

This added control component has the basic form of an irregular dither term that is produced as a linear combination of the outputs ($\bar{M}$) of a high-frequency linear system (Equation 41) whose input is the Kalman-filter innovation variable ($z - H\hat{x}$). In specific cases, moreover, this dither can become an even more significant component of the optimal law approximation, the result of particular order-of-magnitude properties beyond what was assumed in establishing the more general results above. A common example is that of regulating the position of a scalar undamped second-order system when only the position is measured and only the velocity is affected by the plant noise. This means that

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ G = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ H = [1 \vdots 0],\ Q = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix},\ A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

in the notation above. In that case, it can be shown that $\tau = 1/\sqrt{m}$ and that optimal control law approximation of the preceding paragraph is actually accurate except for terms of order h^2 as long as $mh\tau \ll 1$. The dither term is still of order $h^2m^2\tau^{5/2}$ in this case so it will even be large compared to h under these conditions if $m^{-1/2} \gg h \gg m^{-3/4}$. Thus, the initial parameter uncertainty can be large enough in such a case without being too large to destroy the formal validity of the result that the dither control is the next level of asymptotic approximation to the optimal control law after the "linear-quadratic-Gaussian" approximation.

Figure 1 shows the numerical significance of adding such a dither control to an adaptive missile autopilot (attitude control law), which is an example of this particular type of control problem with two parameters altogether. These parameters approximate the effects of missile speed and altitude. The details of this example are described in Reference 11 with the unfortunate exception of an error in computing the dither control according to Equations 37 through 41 and the approximation of $\mathcal{E}[\exp(\theta_i)]$ by $\exp[\mathcal{E}(\theta_i)]$ instead of $\exp[\mathcal{E}(\theta_i) + \frac{1}{2}\mathrm{var}(\theta_i)]$. Briefly, it is a case of planar motion, and the objective is to make a missile's acceleration lateral to its flight path match an arbitrary "commanded" value (which typically varies with time). This lateral acceleration is produced primarily by the body lift that results from orienting the missile at an angle to its flight path, and this orientation is controlled in turn by tail fin deflections. The controller measures actual and commanded lateral acceleration, and selects the current tail fin deflection. The optimization criterion used for the control law is one for which the optimal control makes the acceleration track the commanded value as a critically damped second-order system under idealized conditions (missile dynamics obey a simplified model, and parameter values are known precisely). Figure 1 shows the result of using the indicated control laws in a more realistic computer simulation of the missile and aerodynamics. The commanded acceleration in each case here is the indicated series of 1-sec steps. This is followed closely under the nominal flight condition (Mach 2 at 20,000 ft), both with and without the dither control. At Mach 1.5 and 60,000 ft., however, adding the dither control caused the autopilot to adapt more quickly to this nonnominal flight condition (i.e., to nonzero values of the parameters). The parameter changes corresponding to this difference in flight condition are actually far beyond what has been justified theoretically for using this dither to approximate an optimal control law more accurately. Instead, they correspond to prior parameter variances that, to test the robustness of this approximation, were made as large as would allow the autopilot response to remain empirically reasonable at the nominal flight conditions.

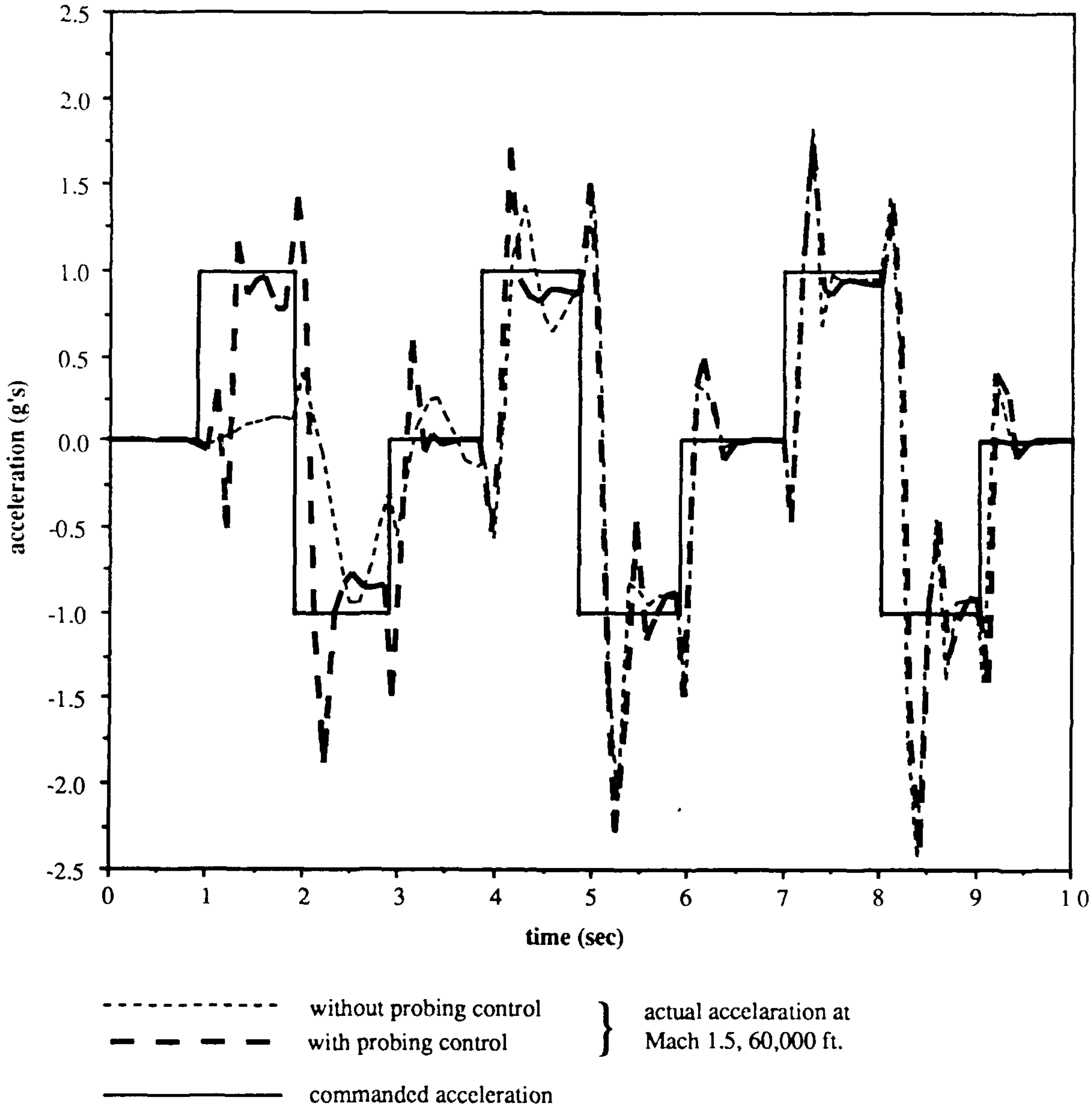

FIGURE 1. Autopilot Response.

Appendix

CONTROL LAW DETAILS

The order-h approximation of the optimal control law for Equations 34 through 36 was developed for the most part in Reference 10. Including the remaining terms in that development shows that this approximation in the notation of Equations 34 through 36 is

$$u = -B^{-1}(\bar{Z}^T + \bar{G}^T S)\hat{y} + B^{-1}\ \mathrm{tr}\{[2(S\bar{G} + \bar{Z})B^{-1}\Psi' - \Upsilon - \Theta S$$
$$-S\Theta^T - (S\bar{G} + \bar{Z})B^{-1}\Xi B^{-1}(\bar{Z}^T + \bar{G}^T S) + (\Pi\bar{G})'']\hat{y}\,\hat{y}^T\}$$
$$-B^{-1}\{c + \bar{G}^T\phi + \mathrm{tr}[\Upsilon P + (S + Y)(P\Theta + \Theta^T P) + P\bar{H}^T\bar{R}^{-1}\bar{H}PY\},$$

where $\hat{y}$ and P are generated from the incoming measurements z by

$$\dot{\hat{y}} = (\bar{F} + \Theta' u)\hat{y} + (\bar{G} + \Theta^T\hat{y})u + \mathrm{tr}[\Delta(\hat{y}\hat{y}^T + P)]$$
$$+ P(\bar{H} + 2\bar{H}\Omega'' u + 2\Phi''\hat{y})^T\bar{R}^{-1}(z - \hat{z});\ \hat{y}(t_0) = \begin{bmatrix} \mathcal{E}[x(t_0)] \\ 0 \end{bmatrix}$$

and

$$\dot{P} = (\bar{F} + 2\Delta''\hat{y} + 2\Theta' u)P + P(\bar{F} + 2\Delta''\hat{y} + 2\Theta' u)^T + \bar{Q}$$
$$-P(\bar{H} + 2\bar{H}\Omega'' u + 2\Phi''\hat{y})^T\bar{R}^{-1}(\bar{H} + 2\bar{H}\Omega'' u + 2\Phi''\hat{y})P$$
$$+2[\Lambda\bar{H}^T + (P\Phi P)']\bar{R}^{-1}(z - \hat{z});\ P(t_0) = \begin{bmatrix} \mathrm{var}[x(t_0)] & 0 \\ 0 & \mathrm{var}(\theta) \end{bmatrix}$$

and where

$$\hat{z} = \bar{H}\hat{y} + \mathrm{tr}[\Phi(\hat{y}\hat{y}^T + P)]$$

and $\Lambda(t)$, $S(t)$, $Y(t)$, $\Pi(t)$, and $\phi(t)$ are the solutions of the differential equation system

$$\dot{V} = \bar{F}V + V\bar{F}^T + \bar{Q} - V\bar{H}^T\bar{R}^{-1}\bar{H}V;\ V(t_0) = P(t_0)$$

$$\dot{\Lambda} = \Gamma + \Gamma' + \Gamma''; \; \Gamma = (\bar{F} - V\bar{H}^T\bar{R}^{-1}\bar{H})\Lambda + V\Lambda V - (V\Phi V)'\bar{R}^{-1}\bar{H}V; \; \Lambda(t_0) = 0$$

$$-\dot{S} = S\bar{F} + \bar{F}^T S + \bar{A} - (S\bar{G} + \bar{Z})B^{-1}(\bar{Z}^T + \bar{G}^T S); \; S(t_f) = \bar{S}$$

$$-\dot{Y} = Y(\bar{F} - V\bar{H}^T\bar{R}^{-1}\bar{H}) + (\bar{F}^T - \bar{H}^T\bar{R}^{-1}\bar{H}V)Y + (S\bar{G} + \bar{Z})B^{-1}(\bar{Z}^T + \bar{G}^T S); \; Y(t_f) = 0$$

$$\begin{aligned} -\dot{\Pi} = \Sigma - [(S\bar{G} + \bar{Z})B^{-1}\Xi B^{-1}(\bar{Z}^T + \bar{G}^T S)]'B^{-1}(\bar{Z}^T + \bar{G}^T S) \\ +\Gamma + \Gamma' + \Gamma''; \; \Gamma = \Pi[\bar{F} - \bar{G}B^{-1}(\bar{Z}^T + \bar{G}^T S)] + \Delta' S \\ -S\Theta' B^{-1}(\bar{Z}^T + \bar{G}^T S) - (S\bar{G} + \bar{Z})B^{-1}\Theta'^T S - \Upsilon' \bar{A}^T \\ +(S\bar{G} + \bar{Z})B^{-1}(\Psi + \Psi'')B^{-1}(\bar{Z}^T + \bar{G}^T S); \; \Pi(t_f) = \Pi_f \end{aligned}$$

$$\begin{aligned} -\dot{\phi} = [\bar{F}^T - (S\bar{G} + \bar{Z})B^{-1}G^T]\phi - (S\bar{C} + \bar{Z})B^{-1}\{c + \mathrm{tr}[\Upsilon V \\ +(S + Y)(V\Theta + \Theta^T V) - V(\Omega\bar{H}^T\bar{R}^{-1}\bar{H} + \bar{H}^T\bar{R}^{-1}\bar{H}\Omega^T)VY]\} \\ +S \; \mathrm{tr}(V\Delta) + \mathrm{tr}[V\Sigma + V\bar{H}^T\bar{R}^{-1}\bar{H}V\Pi + (S + Y)(V\Delta' + \Delta''V) \\ -V(\bar{H}^T\bar{R}^{-1}\Phi'' + \Phi'\bar{R}^{-1}\bar{H})VY]; \; \phi(t_f) = \mathrm{tr}[\Pi_f V(t_f)]. \end{aligned}$$

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